II. The Formalism of Derived Categories

II.1 Triangulated Categories

Let A be an abelian category. The derived category $D(A)$ of A is a quotient category of the abelian category $Kom(A)$ of complexes over A. The quotient category is defined by making quasiisomorphisms into isomorphisms and this allows to identify complexes with their resolutions. Recall, that a complex map $K' \to K$ is a quasiisomorphism, if the induced cohomology morphisms $H^\bullet(K') \to H^\bullet(K)$ are isomorphisms in all degrees. However, by taking this localization of the category $Kom(A)$, the notion of (short) exact sequences of complexes no longer exists and has to be replaced by the notion of distinguished triangles, which itself derives from the concept of mapping cones. The derived category thus becomes a triangulated category, a notion first introduced by Verdier in [Ver], [SGA4$\frac{1}{2}$].

Suppose D is an additive category with an additive automorphism of categories $\mathcal{T} : D \to D$, which is called the translation functor. By abuse of notation we write $X \in D$ to indicate that X is object of a category D. For $n \in \mathbb{Z}$ we usually write $\mathcal{T}^n(X) = X[n]$ resp. $\mathcal{T}^n(f) = f[n]$ both for objects X and morphisms f in D.

A triangle $T = (X, Y, Z, u, v, w)$ in D is a diagram in D

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]\,.$$

Instead of writing such a diagram, we often use the abbreviated abusive way of writing (X, Y, Z) or $(X, Y, Z, *, *, *)$, if the underlying morphisms are understood from the context.

A morphism (f, g, h) between triangles is a commutative diagram

$$\begin{array}{ccccccc} X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1]\,. \end{array}$$

If

$$T = (X, Y, Z, u, v, w)$$

is a triangle, we call

$$rot(T) = (Z[-1], X, Y, -w[-1], u, v)$$

the rotated triangle. If $(f, g, h) : T' \to T$ is a morphism between triangles, then $rot(f, g, h) = (h[-1], f, g)$ is a morphism between the rotated triangles $rot(f, g, h) : rot(T') \to rot(T)$.

Definition 1.1 *A triangulated category is an additive category D with a translation functor $\mathcal{T}$ and a class of distinguished triangles satisfying the following axioms TR1, TR2, TR3 and TR4:*

TR1 (Rotation)

a) *A triangle T is distinguished if and only if its rotated triangle $rot(T)$ is distinguished.*
b) *Triangles isomorphic to distinguished triangles are distinguished.*

Especially $(id, -id, id)^* rot^3(T) = (X[-1], Y[-1], Z[-1], u[-1], v[-1], -w[-1])$ is distinguished, if $T = (X, Y, Z, u, v, w)$ is distinguished. This will be relevant for axiom TR4.

TR2 (Existence of cones)

a) *Any morphism $u : X \to Y$ in D can be completed (not necessarily uniquely) to a distinguished triangle (X, Y, Z, u, v, w). Any such object Z will be called a cone or mapping cone for $u : X \to Y$.*
b) *The triangle $(X, X, 0, id_X, 0, 0)$ is distinguished.*

TR3 (Morphisms). Any commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{u'} & Y' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{u} & Y \end{array}$$

can be extended (not necessarily uniquely) to a morphism (f, g, h) between given distinguished triangles (X', Y', Z', u', v', w') and (X, Y, Z, u, v, w).

Due to axiom TR1 one also has versions of the axioms TR2 and TR3, which are obtained by applying TR2 and TR3 to rotated triangles. Altogether this already has a number of consequences. The most important is the long exact Hom-sequence stated in Theorem II.1.3. We first discuss these consequences before we formulate the last axiom TR4 of triangulated categories.

Remark 1.2 Let (X, Y, Z, u, v, w) be a distinguished triangle. Then $v \circ u = 0$ holds. Use Axiom TR2b and Axiom TR3 with $(f, g) = (id_X, u)$ to deduce this from

$$\begin{array}{ccccccc} X & \xrightarrow{id_X} & X & \xrightarrow{0} & 0 & \xrightarrow{0} & X[1] \\ \downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \exists h} & & \downarrow{\scriptstyle id_X[1]} \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1]\,. \end{array}$$

Furthermore, any morphism $g : A \to Y$ satisfying $v \circ g = 0$ factors over X by a morphism $f : A \to X$. For this apply a rotated version of axiom TR3 to the pair $(g, h) = (g, 0)$:

$$\begin{array}{ccccccc} A & \xrightarrow{id} & A & \xrightarrow{0} & 0 & \xrightarrow{0} & A[1] \\ \downarrow{\scriptstyle \exists f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f[1]} \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1]\,. \end{array}$$

By repeated use of the rotation axiom this proves

Theorem 1.3 *Let (X, Y, Z, u, v, w) be a distinguished triangle in D. For any A in D the sequence*

$$\to Hom(A, X[i]) \xrightarrow{u[i]_*} Hom(A, Y[i]) \xrightarrow{v[i]_*} Hom(A, Z[i]) \xrightarrow{w[i]_*} Hom(A, X[i+1]) \xrightarrow{u[i+1]_*}$$

is a long exact sequence of abelian groups. similarly one gets a long exact Hom-sequence

$$\leftarrow Hom(X[i], B) \xleftarrow{u[i]^*} Hom(Y[i], B) \xleftarrow{v[i]^*} Hom(Z[i], B) \xleftarrow{w[i]^*} Hom(X[i+1], B) \xleftarrow{u[i+1]^*}$$

in the first variable.

Immediate consequences of the long exact Hom-sequence are

Corollary 1.4 *If (f, g, h) is a morphism between distinguished triangles and f and g are isomorphisms, then h is an isomorphism.*

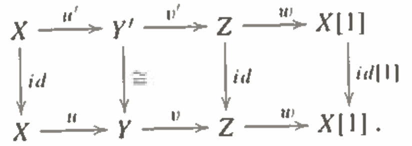
Proof. This follows from the 5-lemma. Applying it to the two $Hom(A, .)$-sequences for (X', Y', Z') and (X, Y, Z) it shows $h_* : Hom(A, Z') \cong Hom(A, Z)$ for all A. Specializing to $A = Z'$, Z gives a right and left inverse to $h : Z' \to Z$. □

Corollary 1.5 *For a given morphism $u : X \to Y$ any two mapping cones Z_u are isomorphic. Two distinguished triangles (X, Y, Z_u, u, v, w) and (X, Y, Z_u, u, v', w') attached to $u : X \to Y$ with the same mapping cone Z_u are related by $v' = h^{-1} \circ v$, $w' = w \circ h$, with an isomorphism h of Z_u.*

Proof. Use Corollary II.1.4 and axiom TR3. □

Conversely, any object Z isomorphic by $h : Z \cong Z_u$ to a cone Z_u of $u : X \to Y$ is itself a cone, since the triangle (X, Y, Z, u, v', w') for $v' = h^{-1} \circ v$ and $w' = w \circ h$ is distinguished by axiom TR1b.

Remark (Rotated Version of Axiom TR2a). Any morphism $w : Z \to X[1]$ can be completed to a distinguished triangle (X, Y, Z, u, v, w). The resulting object Y is called an extension of Z by X attached to $w \in Hom(Z, X[1])$. As a variant of Corollary II.1.4 and II.1.5 any two extensions attached to w are isomorphic.



Corollary 1.6 *If (X, Y, Z, u, v, w) is a distinguished triangle and u is an isomorphism, then $Z \cong 0$. Conversely $Z \cong 0$ implies that u is an isomorphism.*

Proof. The long exact $Hom(Z, .)$-sequence implies $Hom(Z, Z) = 0$ and proves the first statement. The second statement again follows from the long exact Hom-sequence. □

Corollary 1.7 *If (X, Y, Z, u, v, w) is distinguished and $w=0$, then $Y \cong Z \oplus X$ such that u and v correspond to the inclusion resp. projection map.*

Proof. Exercise! First find p with $p \circ u = id_X$, then find i with $id_Y = u \circ p + i \circ v$. Then $v \circ i = id_Z$ and finally $p \circ i = 0$ follow from the injectivity of the dual Hom-sequence at $X[1]$. □

So far we have discussed the axioms TR1-3 of a triangulated category and some trivial implications. We now formulate the remaining axiom TR4 for triangulated categories, the so called octaeder axiom. See also [140], especially for an explanation of the name.

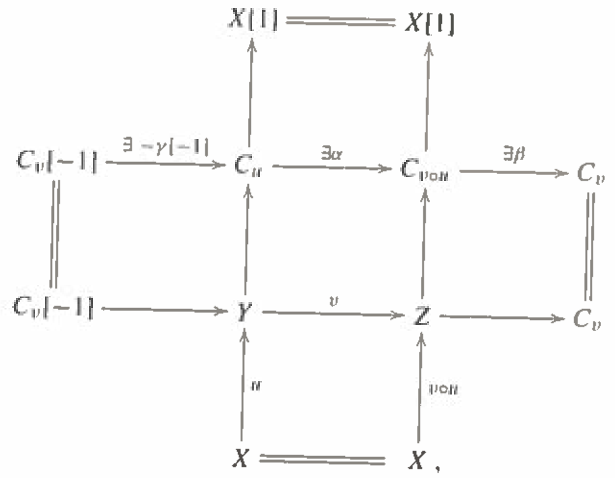
Axiom TR4a (Composition Law for Mapping Cones). Suppose we are given morphisms $u : X \to Y$ and $v : Y \to Z$. For any choice of mapping cones C_u, C_v and $C_{v \circ u}$ with defining distinguished triangles

$$T_u = (X, Y, C_u, u, *, *)$$
$$T_v = (Y, Z, C_v, v, *, *)$$
$$T_{v \circ u} = (X, Z, C_{v \circ u}, v \circ u, *, *)\ ,$$

there exists a distinguished triangle T relating these mapping cones

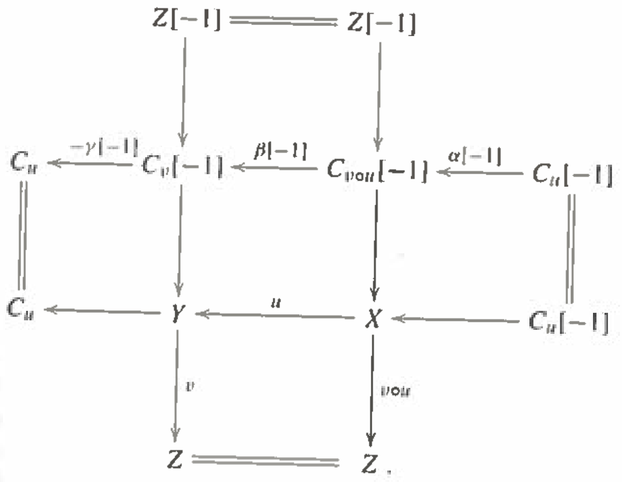
$$T = (C_u, C_{v \circ u}, C_v, \alpha, \beta, \gamma)\ ,$$

which makes the following diagram commute



The two vertical lines of the diagram are defined by the morphisms of the two distinguished triangles T_u, $T_{v \circ u}$, the two horizontal lines by the morphisms of the two rotated distinguished triangles $rot(T_v)$, $rot(T)$. Note, that although we assume the triangles T_u, T_v, $T_{v \circ u}$ to be chosen fixed, we preferred not to give names to all morphisms.

Axiom TR4b. For the full octaeder axiom one adds the further condition of commutativity for the diagram



The two vertical lines of the diagram are defined by the morphisms of the distinguished triangles $rot^2(T_{v \circ u})$, $rot^2(T_v)$, the two horizontal lines by the morphisms of the distinguished triangles $(id, -id, id)^*(rot^3(T))$ and $rot(T_u)$.

In the axioms of a triangulated category there is a certain redundancy: axiom TR3 is a consequence of axiom TR4a and TR4b. Consider maps f, g, u', u as in

TR3. Then axiom TR4a, which is a kind of pushout axiom, applied to $X' \xrightarrow{u'} Y' \xrightarrow{g} Y$ and axiom TR4b, which is a kind of pullback axiom, applied to $X' \xrightarrow{f} X \xrightarrow{u} Y$ together imply TR3. For $k := g \circ u' = u \circ f$ this defines morphisms $\alpha : C_{u'} \to C_k$ respectively $\beta' : C_k \to C_u$. Then $h = \beta' \circ \alpha$ defines a morphism (f, g, h) between the triangles $(X', Y', C_{u'})$ and (X, Y, C_u).

The Derived Categories $D(\mathbf{A})$

The most prominent example for a triangulated category is the derived category $D(\mathbf{A})$ of an abelian category $\mathbf{A}$, as mentioned in the introduction. It is the localization of the category of all complexes $Kom(\mathbf{A})$ over $\mathbf{A}$ by the class of quasiisomorphisms. A short review on this is in the appendix II of [FK], p. 292 and in chapter I of [140] (Hartshorne).

To verify the axioms TR1–4 for derived categories, it is useful to be aware of the fact that the localization functor is not injective on homomorphism groups. In particular homotopic complex maps become equal in the derived category. However this gives some extra freedom. So one can proceed in two steps: first pass to the category $K(\mathbf{A})$, whose morphisms are homotopy classes of complex maps, and then invert quasiisomorphisms. The axioms TR1–4 can be established already on the first level as properties of complexes up to homotopy. To invert quasiisomorphisms in the category $K(\mathbf{A})$ becomes much more convenient, because the class of quasiisomorphisms is a localizing class, i.e. it allows a calculus of fractions in the category $K(\mathbf{A})$. See [FK] A II.1. More details can be found in [Ver], [104], [140].

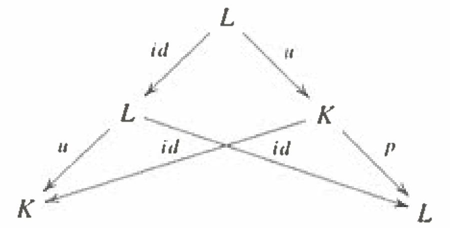
In the same way one can define the derived categories $D^+(\mathbf{A})$, $D^-(\mathbf{A})$, and $D^b(\mathbf{A})$ as the localization of the full subcategory of complexes which are bounded to the left, bounded to the right respectively are bounded. They can be embedded as full subcategories into $D(\mathbf{A})$. In particular for a morphism f in $Hom_{D^*(\mathbf{A})}(K, L)$ for $* \in \{+, -, b\}$ there exist quasiisomorphisms s, t, such that $f \circ s$ respectively $t \circ f$ are homotopic to complex maps.

We remark that the construction of $D(\mathbf{A})$ by localization gives set theoretic problems unless $\mathbf{A}$ is a small category or belongs to some given universe, since a priori $Hom_{D(\mathbf{A})}(X, Y)$ is not a set. However, let the category $\mathbf{A}$ satisfy the Grothendieck Tôhoku axioms ([113]). Then every – may be unbounded – complex K of $Kom(\mathbf{A})$ has a right resolution by a so called *K–injective* complex. Using this fact we can conclude that $D(\mathbf{A})$ is equivalent to the full subcategory of $Kom(\mathbf{A})$ given by all K–injective complexes. See [Spa], [Tar]. This implies, that $Hom_{D(\mathbf{A})}(X, Y)$ is a set in the case of an abelian Grothendieck category $\mathbf{A}$. See also the following remark for the case of the subcategory $D^+(\mathbf{A})$ of $D(\mathbf{A})$.

Remark 1.8 If K, L are in $D^+(\mathbf{A})$ and L is injective, i.e. all components L^ν of the complex L are injective objects of $\mathbf{A}$, then any morphism f in $Hom_{D^+(\mathbf{A})}(K, L)$ is represented by a complex map $p : K \to L$. The localizing property of quasiisomorphisms, which is formulated in [FK] A II 1(2), allows to reduce the proof of this statement to the case, where f is the inverse of a quasiisomorphism $u : L \to K$

(using fractions with left denominators we find u such that $u \circ f$ is a complex map). For the quasiisomorphism u it is enough to show, that there exists a complex map $p : K \to L$ which is a left inverse of u up to homotopy. Namely then (using calculus of fractions with right denominators) the morphism f and the complex map p

or $f : K \xleftarrow{u} L \xrightarrow{id} L$ and $p : K \xleftarrow{id} K \xrightarrow{p} L$ coincide in the derived category, since we have the commutative diagram



To construct p let $C = C_u$ be the cone in the category $K^+(\mathbf{A})$. Then we have a distinguished triangle $(L, K, C, u, *, *)$ and C is acyclic and bounded below. Using Corollary II.1.7 one reduces the construction of a homotopy left inverse p for u to the following fact: If C is acyclic and I is injective ($I = L[1]$) and both C and I are bounded from below, then any complex map $C \to I$ is homotopic to the zero map. This homotopy is constructed inductively using the extension property for injective objects. See e.g. [104], p. 180. So for $K, L \in D^+(\mathbf{A})$ and L injective, the natural map

$$Hom_{K^+(\mathbf{A})}(K, L) \to Hom_{D^+(\mathbf{A})}(K, L)$$

is surjective. It is also injective, since $u \circ f = 0$ for $f \in Hom_{K^+(\mathbf{A})}(K, L)$ and a quasiisomorphism $u : K' \to K$ implies $f = 0$. In fact $u^* : Hom_{K^+(\mathbf{A})}(K, L) \to Hom_{K^+(\mathbf{A})}(K', L)$ is injective. Since $Hom_{K^+(\mathbf{A})}(C_u, L)$ vanishes for the acyclic cone C_u of u, this follows from the long exact Hom-sequence for the triangulated category $K^+(\mathbf{A})$. Therefore the Hom-groups in the derived categories $D^+(\mathbf{A})$ and $D^b(\mathbf{A})$ can be computed in terms of the homotopy category $K(\mathbf{A})$ using injective resolutions in the second variable. This is convenient, provided the abelian category $\mathbf{A}$ has enough injective objects. See Verdier [SGA4$\frac{1}{2}$], p. 299 for further information.

Other examples of triangulated categories can be found in [104]. We will be mainly interested in the triangulated categories $D(X) = D^b_c(X, \overline{\mathbb{Q}}_l)$ for finitely generated schemes X over a finite or algebraically closed field. These categories are obtained as certain limits of derived categories. For further details on these categories the reader is referred to Appendix A and the corresponding section of this chapter.

Remark 1.9 The diagram of axiom TR4b is formally obtained from the diagram of axiom TR4a by replacing the direction of arrows (and renaming). This implies, that the notion of triangulated category is self dual: If D is a triangulated category, then also the opposite category D^{opp}, obtained by inverting arrows with the induced translation functor and induced distinguished triangles, is triangulated. Later we will use this in the proof of Corollary II.4.2. Nevertheless we mention, that the only information added by TR4b to TR4a is the commutativity of the middle square of the TR4b diagram. In other words TR4b is, modulo TR4a, equivalent to either one of the two statements:

TR4b′: $(u, id_Z, \beta) : T_{v\circ u} \to T_v$ is a morphism of distinguished triangles.
TR4b″: The two hidden ways, to go in the diagram of axiom TR4a over the upper right corner from $C_{v\circ u}$ to $Y[1]$, *anticommute*.

More precisely, Axiom TR4b″ states

$$(-i[-1]) \circ \beta = -(-u[1]) \circ k\,.$$

if $T_v = (*, *, *, v, *, i)$ and $T_{v\circ u} = (*, *, *, v \circ u, j, k)$ are the triangles chosen.

Remark 1.10 The isomorphism class of the distinguished triangle T, whose existence is imposed by axiom TR4, is determined (up to isomorphism) by the three cones $C_u, C_v, C_{v\circ u}$ and the morphism γ, according to Corollary II.1.5. However $-\gamma[-1]$ and therefore also γ is uniquely determined by the commutativity of the left square of diagram TR4a – as the composite of the given maps $C_v[-1] \to Y$ and $Y \to C_u$, appearing in the two distinguished triangles T_v and T_u. Thus the triangles $T_u, T_v, T_{v\circ u}$ determine T up to isomorphism.

II.2 Abstract Truncations

In the derived category $D(A)$ of an abelian category A one has full subcategories $D(A)^{\leq n}$ and $D(A)^{\geq m}$, consisting complexes with vanishing cohomology in degrees strictly larger than n resp. strictly smaller than m. By the process of truncation, a given complex can be split into two complexes, one of them in $D(A)^{\leq 0}$ and the other in $D(A)^{\geq 1}$. This has an abstract analog in an arbitrary triangulated category, motivated by the theory of D-modules and the Riemann-Hilbert correspondence. The notion of t-structures was first introduced in [BBD], inspired by the non obvious t-structures underlying perverse sheaves respectively holonomic D-modules with regular singularities.

Definition 2.1 *A t-structure in a triangulated category D consists of two strictly full subcategories $D^{\leq 0}$ and $D^{\geq 0}$ of D, such that with the definitions $D^{\leq n} = D^{\leq 0}[-n]$ and $D^{\geq n} = D^{\geq 0}[-n]$ we have*

(i) $Hom(D^{\leq 0}, D^{\geq 1}) = 0$.
(ii) $D^{\leq 0} \subset D^{\leq 1}$ and $D^{\geq 1} \subset D^{\geq 0}$.
(iii) For every object E in D there exists a distinguished triangle (A, E, B) with $A \in D^{\leq 0}$ and $B \in D^{\geq 1}$.

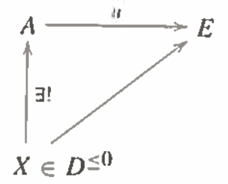
D is said to be bounded with respect to the t-structure, if every object of D is contained in some $D^{\geq a}$ and some $D^{\leq b}$ for certain integers a, b.

Truncation. Suppose we are given a t-structure.

In the situation of II.2.1(iii) we have $B \in D^{\geq 1}$ and $B[-1] \in D^{\geq 1}[-1] \subset D^{\geq 1}$ by property II.2.1(ii). Therefore the long exact Hom-sequence II.1.3 attached to the distinguished triangle $(A, E, B, u, *, *)$ together with the vanishing property II.2.1(i) of t-structures implies

$$u_* : Hom(X, A) \cong Hom(X, E) \quad , \quad \text{for } X \in D^{\leq 0}\,.$$

This fundamental fact has the striking consequence that $u : A \to E$ is a universal morphism from $D^{\leq 0}$ to the given object $E \in D$. Every morphism from some $X \in D^{\leq 0}$ to E factors in a unique way over the morphism u



This universal property characterizes the pair (A, u) uniquely up to isomorphism. We therefore write $A = \tau_{\leq 0}(E)$. The assignment $\tau_{\leq 0}$ is functorial in E. For every morphism $f : E' \to E$, the composite $f \circ u' : \tau_{\leq 0}(E') \to E$ factors through $u : \tau_{\leq 0}(E) \to E$ by a unique morphism $\tau_{\leq 0}(f)$. In other words:

$$\tau_{\leq 0} : D \to D^{\leq 0}$$

defines a functor, which is right adjoint to the inclusion functor of $D^{\leq 0}$ into D. The isomorphism u_* established above turns out to be the adjunction isomorphism.

Similarly, the assignment $E \mapsto B$ defines a functor $\tau_{\geq 1} : D \to D^{\geq 1}$, which is left adjoint to the inclusion of $D^{\geq 1} \subset D$. We will assume, after making some choices, that the functors $\tau_{\leq 0}, \tau_{\geq 1}$ are fixed from now on.

Resume. From the discussion of t-structures so far we see, that the distinguished truncation triangle of property (iii) for t-structures has now become the unique distinguished triangle

iii)′ $$\tau_{\leq 0}(E) \xrightarrow{ad_{\leq 0}} E \xrightarrow{ad_{\geq 1}} \tau_{\geq 1}(E) \to \tau_{\leq 0}(E)[1]\,.$$

The first two morphisms have become adjunction maps. They uniquely determine the third map of the triangle. For this recall Corollary II.1.5 and the fact, that $h = id_{\tau_{\geq 1}(E)}$ is the unique morphism $h : \tau_{\geq 1}(E) \to \tau_{\geq 1}(E)$ with the property $h \circ ad_{\geq 1} = ad_{\geq 1}$ because of the adjunction formula

$$Hom(E, Y) = Hom(\tau_{\geq 1}(E), Y) \quad , \quad \text{for } Y \in D^{\geq 1}\,.$$

Similarly the isomorphism u_* from above gives the adjunction formula

$$Hom(X, \tau_{\leq 0}(E)) = Hom(X, E) \quad , \quad \text{for } X \in D^{\leq 0} .$$

Properties of the truncation functors. Recursively define $\tau_{\leq n}$ for $n \in \mathbb{Z}$ by

$$(\tau_{\leq n+1}(X))[1] = \tau_{\leq n}(X[1])$$

or $\tau_{\leq n}(X) = (\tau_{\leq 0}(X[n]))[-n]$. Then $\tau_{\leq n}$ is right adjoint to the inclusion functor $D^{\leq n} \subset D$. Therefore, by the obvious inclusion properties of the underlying categories

$$\tau_{\leq m} \circ \tau_{\leq n} = \tau_{\leq m} \qquad \text{for} \quad m \leq n .$$

Lemma 2.2 (Orthogonality) *For objects $E \in D$ the following statements are equivalent*

(i) E is in $D^{\geq n+1}$.
(ii) $Hom(D^{\leq n}, E) = 0$.

Proof. One easily reduces to $n = 0$. Then one direction is the statement of property (i) for t-structures. For the converse direction (ii) $\Rightarrow$ (i) it is enough to show $\tau_{\leq 0}(E) = 0$ by Corollary II.1.6. But $\tau_{\leq 0}(E) = 0$ follows from the adjunction isomorphism $Hom(\tau_{\leq 0}(E), \tau_{\leq 0}(E)) = Hom(\tau_{\leq 0}(E), E) \subset Hom(D^{\leq 0}, E) = 0$. □

We also write $\tau_{>n}$ for $\tau_{\geq n+1}$ or $\tau_{<n}$ for $\tau_{\leq n-1}$.

Lemma 2.3 (Extensions) *Suppose (X, Y, Z) is a distinguished triangle. Then*

1) If X and Z are in $D^{\leq n}$, then also Y is in $D^{\leq n}$.
2) If Y is in $D^{\leq n}$ and if X is in $D^{\leq n+1} \supset D^{\leq n}$, then also Z is in $D^{\leq n}$.

Proof. Apply the long exact *Hom*-sequence II.1.3 and the criterion (ii) $\Rightarrow$ (i) of Lemma II.2.2. □

The statements of Lemma II.2.2 and II.2.3 have dual versions. We define

$$(\tau_{\geq m+1}(X))[1] = \tau_{\geq m}(X[1])$$

with $\tau_{\geq n} \circ \tau_{\geq m} = \tau_{\geq n}$ for $m \leq n$. Suppose (X, Y, Z) is distinguished. Again $X, Z \in D^{\geq m}$ implies $Y \in D^{\geq m}$ and $Y \in D^{\geq m}$, $Z \in D^{\geq m-1} \supset D^{\geq m}$ implies $X \in D^{\geq m}$.

Lemma 2.4 (Compatibility) *We have*

$$\tau_{\geq m}(D^{\leq n}) \subset D^{\leq n}$$

and similarly

$$\tau_{\leq n}(D^{\geq m}) \subset D^{\geq m} .$$

For $m > n$ we have $\tau_{\geq m}(D^{\leq n}) = \tau_{\leq n}(D^{\geq m}) = 0$.

Proof. We only proof the first statement. For $n < m$ and $X \in D^{\leq n}$ the natural map $X \to \tau_{\geq m}X$ is zero by II.2.1(i). Therefore the adjunction formula implies $Hom(\tau_{\geq m}X, \tau_{\geq m}X) = 0$, hence $id_{\tau_{\geq m}X} = 0$ or $\tau_{\geq m}X = 0$. So assume $m \leq n$. Consider the distinguished triangle $(\tau_{\leq m-1}X, X, \tau_{\geq m}X)$. For $X \in D^{\leq n}$ and $m \leq n$ we have $\tau_{\leq m-1}(X) \in D^{\leq n-1} \subset D^{\leq n+1}$. The claim now follows from the extension Lemma II.2.3. □

Exercise 2.5 Two t-structures α and β of a triangulated category D which are included in each other in the sense that ${}^{\alpha}D^{\leq n} \subset {}^{\beta}D^{\leq n}$, ${}^{\alpha}D^{\geq m} \subset {}^{\beta}D^{\geq m}$ necessarily coincide.

II.3 The Core of a t-Structure

In the derived category $D(A)$ attached to an abelian category A one can reconstruct A from the canonical t-structure by considering the full subcategory of complexes in $D(A)^{\leq 0} \cap D(A)^{\geq 0}$. This is the category of complexes with vanishing cohomology outside of degree zero. The natural functor $A \to D(A)^{\leq 0} \cap D(A)^{\geq 0}$ defines an equivalence of abelian categories. In general, a t-structure on a triangulated category D defines in a similar way an abelian category A, the core $Core(D) = Core_t(D)$. However, the relationship between D and $D(Core(D))$ is not clear in general[1].

Let D be a triangulated category with a given t-structure $D^{\leq 0}$ and $D^{\geq 0}$. Define the core $Core(D)$ to be the full subcategory $Core(D) = D^{\leq 0} \cap D^{\geq 0}$.

Theorem 3.1 *The core $Core(D) = D^{\leq 0} \cap D^{\geq 0}$ attached to a t-structure of a triangulated category D is an abelian category. A sequence in $Core(D)$*

$$0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$$

is exact if and only if there exists a distinguished triangle (X, Y, Z, u, v, w) in D.

Proof. For a morphism $f : X \to Y$ between objects of $Core(D)$ choose a cone Z_f in D with distinguished triangle (X, Y, Z_f) and put

$$Ker_f := \tau_{\leq 0}(Z_f[-1])$$

$$Koker_f := \tau_{\geq 0}Z_f .$$

Step 1. For $A \in D^{\leq 0}$ we have $Hom(A, \tau_{\leq 0}(Z_f[-1])) \cong Hom(A, Z_f[-1])$, the adjunction isomorphism. Furthermore $Hom(A, Y[-1]) \subset Hom(D^{\leq 0}, D^{\geq 1}) = 0$, and the long exact *Hom*-sequence gives the short exact sequence

[1] For the core of perverse sheaves considered in Chap. III this was clarified by Beilinson [Be1].

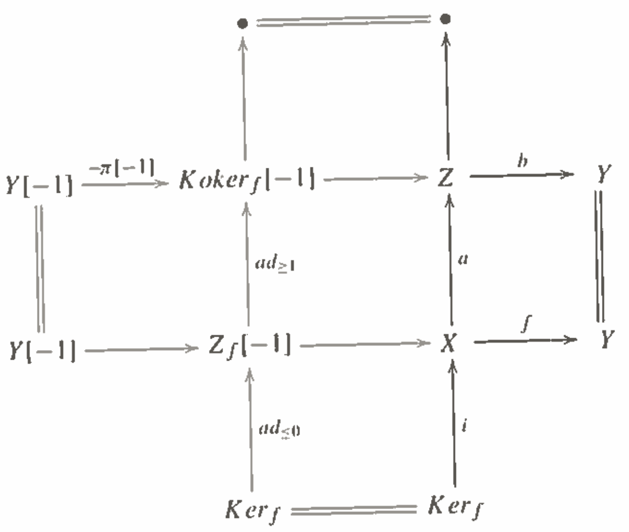
$$0 \to Hom(A, \tau_{\leq 0}(Z_f[-1])) \to Hom(A, X) \to Hom(A, Y) .$$

Thus $i : Ker_f \to X$ with $i : \tau_{\leq 0}(Z_f[-1]) \to Z_f[-1] \to X$ represents the kernel of f in $D^{\leq 0}$. Similarly $\tau_{\geq 0} Z_f$ with $\pi : Y \to Z_f \to \tau_{\geq 0} Z_f = Koker_f$ represents the kokernel of f in $D^{\geq 0}$.

Step 2. For the existence of kernels and kokernels in $Core(D)$ it remains to verify, that Ker_f, $Koker_f$ are objects contained in $Core(D)$. Indeed $Z_f \in D^{\leq 0} \cap D^{\geq -1}$, by the extension Lemma II.2.3 and its dual version. Therefore the compatibility Lemma II.2.4 implies that Ker_f, $Koker_f$ are in $Core(D)$.

Step 3 (Factorization Property). It remains to show, that $f : X \to Y$ decomposes into a composite of $a : X \to Z$ and $b : Z \to Y$ for some $Z \in Core(D)$ with induced isomorphisms $a_* : Z \cong Koker(i : Ker_f \to X)$ and $b_* : Z \cong Ker(\pi : Y \to Koker_f)$.

To find Z and the factorization $f = b \circ a$ consider the octaeder axiom attached to the composition of arrows defining $i : Ker_f \to X$, which gives the following commutative diagram



From $Y \in D^{\geq 0}$ and $Koker_f[-1] \in Core(D)[-1] \subset D^{\geq 1} \subset D^{\geq 0}$, we get $Z \in D^{\geq 0}$ using the extension Lemma II.2.3. Once more Lemma II.2.3, and $X, Ker_f \in Core(D) \subset D^{\leq 0}$, shows $Z \in D^{\leq 0}$. Thus Z is an object of $Core(D)$

$$Z \in Core(D) .$$

Step 4 (Remaining Verifications). The fact that Z is in $Core(D)$ implies, recalling the definition of kernel and kokernel in step 1,

$$b_* : Z \cong Ker(\pi : Y \to Koker_f)$$

$$a_* : Z \cong Koker(i : Ker_f \to X) .$$

The second isomorphism a_* comes from the distinguished triangle (Ker_f, X, Z) with Z as a cone for the morphism $i : Ker_f \to X$. Therefore $Koker_i = \tau_{\geq 0}(Z) = Z$.

The first isomorphism b_* comes from the triangle $(Y[-1], Koker_f[-1], Z)$, which makes Z into a cone for the morphism $\pi[-1] : Y[-1] \to Koker_f[-1]$. In fact, the commutativity of the left square of the diagram above shows that its upper map is $-\pi[-1]$ and we are allowed to modify the triangle by an isomorphic one. Therefore we get $Ker_\pi = \tau_{\leq 0}(Z[1][-1]) = Z$.

Step 5. The additivity axiom finally making $Core(D)$ into an abelian category is clearly satisfied, as $Core(D)$ is closed under extensions by Lemma II.2.3. □

Lemma 3.2 *For X, Z in $Core(D)$ we have*

$$Ext^1_{Core(D)}(Z, X) = Hom(Z, X[1]) .$$

Proof. Note that any distinguished triangle (X, Y, Z) satisfies $Y \in Core(D)$ by the assumptions and Lemma II.2.3 and its dual. Therefore the proof follows from the definition of $Core(D)$ and the remark after Corollary II.1.5. The identifying isomorphism is described as follows: Any extension of Z by X in the abelian category $Core(D)$ corresponds to an isomorphism class $[E]$ of an exact sequence in $Core(D)$

$$E : \quad 0 \to X \to Y \to Z \to 0 .$$

By definition, such an exact sequence in the core arises from a distinguished triangle (X, Y, Z, u, v, w) in D. The isomorphism class $[E]$ of it is uniquely determined by $w \in Hom(Z, X[1])$. See Corollary II.1.5 and the remark thereafter. □

The corresponding statement for higher Ext-groups is satisfied for the derived categories $D(A)$ with their canonical t-structure. It might not be satisfied in general.

3.3 Thick Subcategories. A full triangulated subcategory C of a triangulated category D is called a **thick subcategory**, if objects $X, Y \in D$ must lie in C, provided there exists a morphism $f : X \to Y$ such that

1) f factors over an object in C
2) f has a cone $Cone_f \in C$.

If D is a triangulated category and if C is a thick triangulated subcategory, then the quotient category D/C is a triangulated category. See [SGA4$\frac{1}{2}$], p. 276ff.

Now let **A** be an abelian category. A **Serre subcategory B** or thick subcategory of the abelian category **A** is by definition a full subcategory, which is closed under taking subquotients and extensions. The quotient category **A**/**B** exists as an abelian category. It is obtained by inverting all morphisms, whose kernel and cokernel is in **B**. This class of morphisms allows a calculus of fraction. The quotient functor $\mathbf{A} \to \mathbf{A}/\mathbf{B}$ is an exact functor and maps objects of **B** to the zero object. Any functor from **A** into an abelian category with this property factorizes over the quotient functor $\mathbf{A} \to \mathbf{A}/\mathbf{B}$. See Gabriel [102], p. 364ff.

Examples. Suppose D is a triangulated category with t-structure and core $\mathbf{A} = Core(D)$. Suppose C is a thick subcategory of D, which is stable under the corresponding truncation functor $\tau_{\leq 0}$. Then $\mathbf{B} = C \cap Core(D)$ is a Serre subcategory of the abelian category **A**.

Suppose D is a triangulated category with t-structure. Suppose **B** is a Serre subcategory of $\mathbf{A} = Core(D)$. Then the full subcategory C defined by the objects X in D, whose cohomology objects $H^i(X)$ – as defined in the next section! – are in **B** for all $i \in \mathbb{Z}$, is a thick triangulated subcategory of D stable under the truncation functor $\tau_{\leq 0}$.

Now assume that **A** is a noetherian and artinian abelian category, and suppose **B** is a Serre subcategory of **A**. Then objects X of **A** have a unique maximal subobject X_s isomorphic to an object in **B**, and a unique (up to isomorphism) maximal quotient object X_q isomorphic to an object in **B**. The quotient ${}_rX = X/X_s$ is left reduced – i.e. has no nontrivial subobjects from **B** – and the kernel $X_r = Ker(X \to X_q)$ is right reduced – i.e. has no nontrivial quotient objects from **B**. The epimorphism $X \to {}_rX$ induces an isomorphism $({}_rX)_r \cong {}_r(X_r)$. For this consider the left vertical morphism of the next diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_r & \longrightarrow & X & \longrightarrow & X_q & \longrightarrow & 0 \\ & & \downarrow \exists! & & \downarrow & & \downarrow \exists! & & \\ 0 & \longrightarrow & ({}_rX)_r & \longrightarrow & {}_rX & \longrightarrow & ({}_rX)_q & \longrightarrow & 0 \end{array}$$

Using the connecting morphism of the snake lemma applied to the diagram, it follows that the cokernel of the left vertical morphism is in **B**. Since $({}_rX)_r$ is right reduced, this cokernel therefore is zero. Thus $X_r \to ({}_rX)_r$ is an epimorphism. The kernel K of this epimorphism is in **B** (a subobject of X_s). Since ${}_rX$ has no subobjects in **B**, also $({}_rX)_r$ has no subobjects in **B**. Therefore K is the maximal subobject of X_r in **B**. Thus $({}_rX)_r \cong {}_r(X_r)$.

The object $({}_rX)_r \cong {}_r(X_r)$ is a reduced object, by which we mean that it has neither a nontrivial subobject nor a nontrivial quotient isomorphic to an object of **B**. Any object X becomes isomorphic in the quotient category $\mathbf{A}/\mathbf{B}$ to its reduced subquotient $({}_rX)_r$. The quotient category $\mathbf{A}/\mathbf{B}$ can be described up to equivalence in the following way: It has the same objects as **A**, but modified homomorphism groups such that – see [102]: objects X, Y of **A** become isomorphic in $\mathbf{A}/\mathbf{B}$ if and only if their reduced objects are isomorphic in **A**. For reduced objects X, Y one has the equality

$$Hom_{\mathbf{A}/\mathbf{B}}(X, Y) = Hom_{\mathbf{A}}(X, Y)\,.$$

Hence $\mathbf{A}/\mathbf{B}$ is equivalent to the full subcategory of **A** defined by the reduced objects (this however is in general not an abelian subcategory of **A**).

If Y is left reduced and X is arbitrary, then $Hom_{\mathbf{A}/\mathbf{B}}(X, Y) = Hom_{\mathbf{A}}(X_r, Y)$. The reader is advised to describe the general composition law!

II.4 The Cohomology Functors

It is a remarkable fact, that the derived category combines homological and cohomological properties. This property of derived categories carries over to triangulated categories with t-structure.

Let D be a triangulated category with a t-structure. We define

$$H^0(X) = \tau_{\leq 0}\tau_{\geq 0}X \in Core(D)\,.$$

More generally we define for $n \in \mathbb{Z}$ the **n-th cohomology functors**

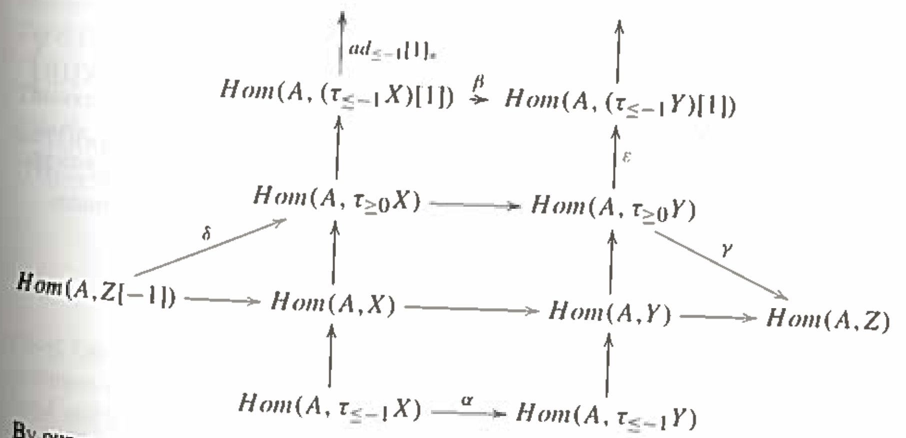
$$H^n : D \to Core(D)$$

by $H^n(X) = H^0(X[n])$; and similar $H^n(u) = \tau_{\leq 0}\tau_{\geq 0}(u[n])$ for morphisms u. Note that

$$H^n(X)[-n] = \tau_{\leq n}\tau_{\geq n}X\,.$$

We will prove below, that for these functors and for a distinguished triangle (X, Y, Z) in the triangulated category D there exists a long exact cohomology sequence in the abelian category $Core(D)$.

To begin with let us assume $Z = \tau_{\geq 0}Z \in D^{\geq 0}$: Then $Z[-1] \in D^{\geq 1}$ is contained in $D^{\geq 0}$. Note that $\tau_{\leq 0}(Z[-1]) = 0$ by II.2.4. For a distinguished triangle (X, Y, Z, u, v, w) and any A in D we now get the following commutative diagram with exact columns and the exact horizontal row, using the long exact Hom-sequences (Theorem II.1.3)



By our assumption on Z we get

a) γ as in the diagram exists! It is the map induced from

$$c : \tau_{\geq 0}Y \to Z\,,$$

where c is obtained from the factorization $Y \to \tau_{\geq 0}Y \to \tau_{\geq 0}Z = Z$.

b) $\tau_{\leq -1}X \xrightarrow{\sim} \tau_{\leq -1}Y$, thus α and β are isomorphisms.

The second statement follows by adjunction from $Hom(A, X) \cong Hom(A, Y)$ for all $A \in D^{\leq -1}$, using Theorem II.1.3 and $Hom(A, Z) = Hom(A, Z[-1]) = 0$ for $A \in D^{\leq -1}$ by II.2.1(i).

Claim. For $Z = \tau_{\geq 0}Z$ the sequence

$$(S) \qquad Hom(A, Z[-1]) \xrightarrow{\delta} Hom(A, \tau_{\geq 0}X) \to Hom(A, \tau_{\geq 0}Y) \xrightarrow{\gamma} Hom(A, \tau_{\geq 0}Z)$$

is exact for any object A of the triangulated category D.

By an easy diagram chase – which is left to the reader – this is reduced to the exactness of the horizontal and the two vertical lines of the diagram. Exactness at $Hom(A, \tau_{\geq 0}X)$ follows from observation b) above. Exactness at $Hom(A, \tau_{\geq 0}Y)$ uses

c) The following holds

$$\beta^{-1}\varepsilon\big(Ker(\gamma)\big) \subset Ker\big(ad_{\leq -1}[1]_*\big)\,.$$

Proof of c). Apply axiom TR3 of triangulated categories, which provides a morphism f and a commutative diagram (e inducing ε, c inducing γ)

$$\begin{array}{ccccccc}
Y & \longrightarrow & \tau_{\geq 0}Y & \xrightarrow{e} & (\tau_{\leq -1}Y)[1] & \xrightarrow{ad_{\leq -1}[1]} & Y[1] \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \exists f} & & \downarrow{\scriptstyle id} \\
Y & \longrightarrow & Z & \longrightarrow & X[1] & \xrightarrow{u[1]} & Y[1]\,.
\end{array}$$

Note $f = adj \circ \tau_{\leq -2}(f)$. Apply $Hom(A, .)$ to this diagram, to obtain

$$\varepsilon\Big(Ker(\gamma)\Big) \;\subset\; Ker\Big(f_*\Big) := Ker\Big(Hom(A, (\tau_{\leq -1}Y)[1]) \to Hom(A, X[1])\Big)\,.$$

However under the isomorphism β^{-1} induced by the isomorphism $(\tau_{\leq -1}u)[1]$ the subgroup $Ker(f_*)$ maps to the kernel $Ker(ad_{\leq -1}[1]_*)$ of the morphism $ad_{\leq -1}[1] : (\tau_{\leq -1}X)[1] \to X[1]$, if the lower triangle of the following diagram commutes

$$\begin{array}{ccc}
X[1] & \xrightarrow{u[1]} & Y[1] \\
\uparrow{\scriptstyle ad_{\leq -1}[1]} & \nwarrow{\scriptstyle f} & \uparrow{\scriptstyle ad_{\leq -1}[1]} \\
(\tau_{\leq -1}X)[1] & \xrightarrow{(\tau_{\leq -1}u)[1]} & (\tau_{\leq -1}Y)[1]\,.
\end{array}$$

Commutativity of the diagram. The upper triangle and the outer square of this diagram commute by definition. Apply $\tau_{\leq -2}$, which collapses the diagram to its lower map. In particular $\tau_{\leq -2}(f)$ becomes an inverse of the isomorphism $(\tau_{\leq -1}u)[1]$ (fact b)). However f factors over the morphism $\tau_{\leq -2}(f)$ to $(\tau_{\leq -1}X)[1]$ by adjunction. Therefore the lower part of the diagram also commutes. □

Corollary 4.1 *Recall the exact sequence (S) of the last claim, which was proved for objects $Z \in D^{\geq 0}$. If we specialize to A varying in $Core(D)$, then it implies the exact sequence*

$$0 \to H^0(X) \to H^0(Y) \to H^0(Z) \qquad in\ Core(D)\,,$$

using the adjunction isomorphism $Hom(A, B) = Hom(A, \tau_{\leq 0}B)$ for $A \in Core(D)$, $B \in D$.

Corollary 4.2 *Dually we obtain for $H_0(X) = \tau_{\geq 0}\tau_{\leq 0}X \in Core(D)$ and distinguished triangles (X, Y, Z) in D with $X \in D^{\leq 0}$ the following exact sequence*

$$H_0(X) \to H_0(Y) \to H_0(Z) \to 0 \qquad in\ Core(D)\,.$$

The exact sequences II.4.1 and II.4.2 can be fitted together using

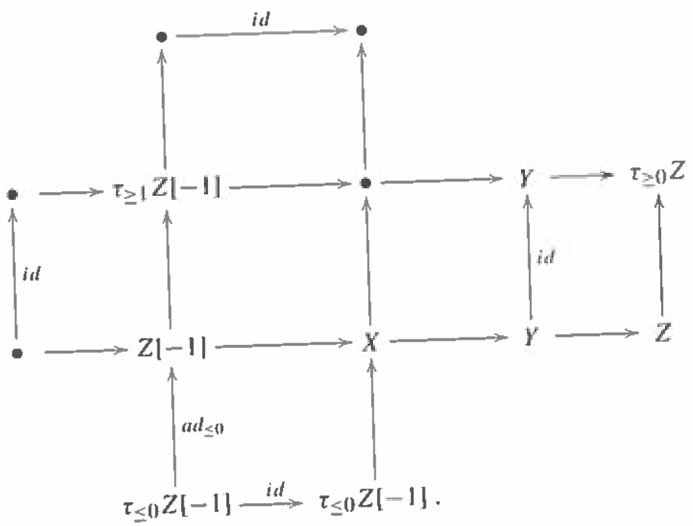
Theorem 4.3 *Let D be a triangulated category with t-structure. For the corresponding functors H_0 and H^0 there exists a functorial isomorphism $\delta : H_0(X) \xrightarrow{\sim} H^0(X)$.*

We postpone the proof of Theorem II.4.3 to the end of this section. Using this theorem, we will tacitly identify $H_0(X) = H^0(X)$. Then the two exact sequences in $Core(D)$, proved above, turn out to be special cases of

Theorem 4.4 *Let D be a triangulated category and let (X, Y, Z) be a distinguished triangle in D. The cohomological functors attached to a t-structure of D induce a long exact cohomology sequence in $Core(D)$*

$$\ldots \to H^{-1}(Z) \to H^0(X) \to H^0(Y) \to H^0(Z) \to H^1(X) \to \ldots\,.$$

Proof. Obviously one gets a complex. By the rotation axiom TR1 it is enough to show exactness at one place, say $H^0(Y)$. The proof of this reduces to the special cases II.4.1 and II.4.2 considered above. Namely apply the functor H^0 to the octaeder diagram TR4a (extended to the right)

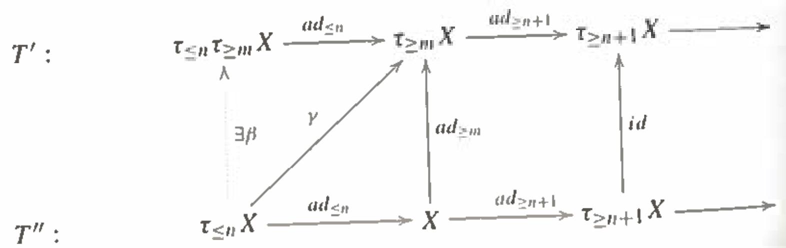


Now exactness at $H^0(Y)$ in the lower row can be deduced by a diagram chase from exactness at $H^0(Y)$ in the upper row. Except the trivial statement $H^0(Z) = H^0(\tau_{\geq 0}Z)$ this uses the fact, that $H^0(X) \to H^0(\bullet)$ is an epimorphism. This follows from II.4.2 for the distinguished triangle $(\tau_{\leq 0}Z[-1], X, \bullet)$ with $\tau_{\leq 0}Z[-1] \in D^{\leq 0}$. Finally exactness at $H^0(Y)$ in the upper row follows from II.4.1 applied to the distinguished triangle $(\bullet, Y, \tau_{\geq 0}Z)$ with $\tau_{\geq 0}Z \in D^{\geq 0}$. □

Exercise. If D is bounded with respect to the t-structure, then $X = 0$ resp. X is in $D^{\leq 0}$ or in $D^{\geq 1}$ iff $H^n(X) = 0$ for all n, resp. for all $n \geq 1$ resp. for all $n \leq 0$.

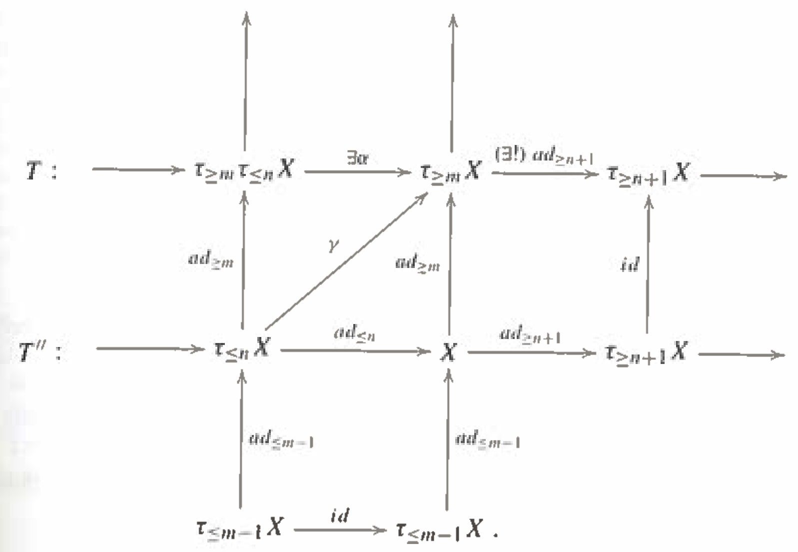
Proof of Theorem II.4.3. Suppose given integers $n \geq m$.

Apply $\tau_{\leq n} \to id \to \tau_{\geq n+1}$ to X and $\tau_{\geq m}X$, for $X \in D$. This gives two distinguished triangles T'' and T' together with a morphism $T'' \to T'$ induced by $(\beta, ad_{\geq m}, id)$ for some β, which is obtained from axiom TR3 by completing the left square of



Here we used $ad_{\geq n+1} \circ ad_{\geq m} = ad_{\geq n+1}$.

On the other hand, starting from the lower square $ad_{\leq n} \circ ad_{\leq m-1} = ad_{\leq m-1}$ the octaeder diagram gives a morphism $T'' \to T$ of distinguished triangles:



Actually the upper morphism in the right square, whose existence is provided by the octaeder axiom, is uniquely determined. It has to be $ad_{\geq n+1}$ by adjunction.

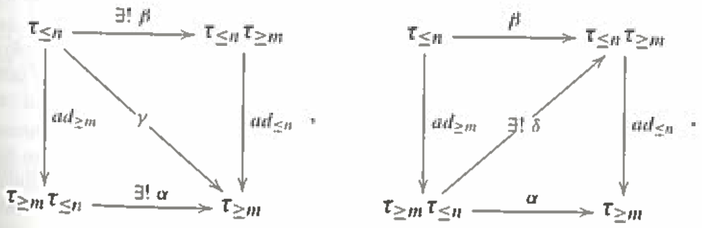
Now compare the distinguished triangles T' and T. Both triangles complete the morphism $ad_{\geq n+1} : \tau_{\geq m}X \to \tau_{\geq n+1}X$. Therefore, by Corollary II.1.4, there exists an isomorphism $(\delta, id, id) : T \xrightarrow{\sim} T'$. In particular we get an isomorphism

$$\delta : \tau_{\geq m}\tau_{\leq n}X \xrightarrow{\sim} \tau_{\leq n}\tau_{\geq m}X .$$

For $m = n = 0$ the existence of this isomorphism proves Theorem II.4.3, except that δ might not be functorial (since a priori it need not be uniquely defined). So the proof of Theorem II.4.3 will be completed by the following consideration.

Functoriality of δ. Put $\gamma = ad_{\geq m} \circ ad_{\leq n}$. This is a natural transformation.

Claim: α, β and δ (as chosen above) make the following two diagrams commutative and this characterizes them uniquely, making them into natural transformations as well:



The proof of the claim is not difficult and we leave the details of its verification to the reader. One has to use the existence of the maps $T'' \to T'$ and $T'' \to T$ defined above (for the convenience of the reader we already have filled in γ into the left upper diagram) and the following sequence of isomorphisms (given by adjunction isomorphisms and Lemma II.2.4 for $m \leq n$)

$$\begin{array}{cccc}
\gamma \in & Hom(\tau_{\leq n}, \tau_{\geq m}) & \xleftarrow{\cong} Hom(\tau_{\leq n}, \tau_{\leq n}\tau_{\geq m}) & \ni \beta \\
& \uparrow \cong & & \\
\alpha \in & Hom(\tau_{\geq m}\tau_{\leq n}, \tau_{\geq m}) & \xleftarrow{\cong} Hom(\tau_{\geq m}\tau_{\leq n}, \tau_{\leq n}\tau_{\geq m}) & \ni \delta\, .
\end{array}$$

The horizontal maps are defined by the composition $\psi \mapsto ad_{\leq n} \circ \psi$, the vertical map by the composition $\phi \mapsto \phi \circ ad_{\geq m}$. Finally $\beta = \delta \circ ad_{\geq m}$ since $ad_{\leq n} \circ \beta = \gamma$ and $ad_{\leq n} \circ \delta \circ ad_{\geq m} = \alpha \circ ad_{\geq m} = \gamma$. □

We get as a

Corollary-Definition 4.5 (Assume $n \geq m$). The natural transformation δ defines an isomorphism $\delta : \tau_{\geq m}\tau_{\leq n} \cong \tau_{\leq n}\tau_{\geq m}$ and we obtain a functor

$$\tau_{[m,n]} = \tau_{\geq m}\tau_{\leq n} \cong \tau_{\leq n}\tau_{\geq m}\, ,$$

which is the identity on $D^{[m,n]} := D^{\leq n} \cap D^{\geq m}$.

Exercise 4.5 Let D be a triangulated category, bounded with respect to some t-structure. Suppose given $K \in Ob(D)$ and $N : K \to K$ a nilpotent morphism in D, i.e. $N^n = 0$ for some integer n. Then the following statements are equivalent:

(i) $K \in Core_t(D)$

(ii) The cone of $N : K \to K$ is in ${}^tD^{[-1,0]}(X)$.

Hint: Show $Cone(N^{2i}) \in {}^tD^{[a,b]}(X)$ if and only if $Cone(N^i) \in {}^tD^{[a,b]}(X)$, using the triangle $(Cone(N^i), Cone(N^{2i}), Cone(N^i))$, to reduce to the case $N = 0$.

II.5 The Triangulated Category $D^b_c(X, \overline{\mathbb{Q}}_l)$

The triangulated category of $\overline{\mathbb{Q}}_l$-sheaves is not a derived category in the original sense. It is obtained as a localization of a projective limit of derived categories, under certain finiteness assumptions. We remind the reader, that the triangulated category of $\overline{\mathbb{Q}}_l$-sheaves has already been used in Chap. I in order to define the Deligne-Fourier transform. It will play an even more important role in the following chapters of the book, in particular for the definition of the category of perverse sheaves. A natural way to construct such a triangulated category of $\overline{\mathbb{Q}}_l$-sheaves is to consider it as the direct limit of triangulated categories of E-sheaves, where E runs over all finite extension fields of the field $\mathbb{Q}_l$. However it is nontrivial, to define such a triangulated category of E-sheaves with good properties. A naive natural candidate to start from would be the derived category of the abelian category of π-adic sheaves introduced in this section. Unfortunately this abelian category of π-adic sheaves does not have sufficiently many injectives or even acyclic objects, in order to define the interesting functors $f_*(-)$ and $\mathcal{H}om(\mathcal{G}, -)$, ... as derived functors in the usual way. The authors of this book do not know

whether there is a mysterious way to define such derived functors with reasonable properties in a direct way on the derived category of π-adic sheaves or the derived category of the abelian category of constructible E-sheaves. On the other hand, Ekedahl [93] has defined a substitute of the derived category of the abelian category of π-adic sheaves. It is a triangulated category with all the good properties, and which allows to define the "derived" functors of all the usual functors in a reasonable way. In fact, Ekedahl's definition is technically rather involved. Fortunately, by the assumptions on the base scheme, which are made throughout this book, the definition considerably simplifies. Certain finiteness theorems then allow to define the desired category by a kind of projective limit, more precisely a projective limit of 2-categories. We start with the notion of π-adic sheaves.

The general reference for the following is [FK], chap. I, §12. See furthermore [SGA4$\frac{1}{2}$], rapport 4.5–4.8 and also Appendix A of this book.

In this section X will denote finitely generated schemes over a finite field or over an algebraically closed field.

Let E be a finite extension field of the field $\mathbb{Q}_l$ of l-adic numbers, let $\mathfrak{o}$ be the valuation ring of E and let π be a generator of the maximal ideal of $\mathfrak{o}$. Let

$$\mathfrak{o}_r = \mathfrak{o}/\pi^r\mathfrak{o} \quad , \quad r \geq 1\, .$$

The prime number l is assumed to be invertible on X.

Excursion on π-Adic Sheaves

We freely use notions and results from [FK] in the following. In particular the notion of the **A-R category (Artin-Rees category)** will be of importance. However it will be necessary to generalize from the l-adic situation considered in [FK] to the π-adic case. The necessary modifications for this generalization are rather obvious. See also Appendix A.

Suppose

$$\mathcal{G} = (\mathcal{G}_r)_{r \geq 1}$$

is a projective system of constructible torsion $\mathfrak{o}$-module sheaves $\mathcal{G}_r$ on X, such that $\pi^r\mathcal{G}_r = 0$. Then $\mathcal{G}_r$ is a sheaf of $\mathfrak{o}_r$-modules.

The system $\mathcal{G}$ is called a **π-adic sheaf**, if the following holds

$$\mathcal{G}_r \otimes_{\mathfrak{o}_r} \mathfrak{o}_s = \mathcal{G}_s \qquad \text{for all} \quad r \geq s\, .$$

1. The projective system $\mathcal{G}$ is called **flat**, if all $\mathcal{G}_r$ have $\mathfrak{o}_r$-free stalks for all geometric points. The projective system $\mathcal{G}$ is called **smooth**[2], if all the sheaves $\mathcal{G}_r$ are locally constant sheaves. For a flat π-adic system $\mathcal{G}$ and integers $1 \leq s \leq r$ one has exact sequences of sheaves

$$0 \to \mathcal{G}_{r-s} \to \mathcal{G}_r \to \mathcal{G}_s \to 0\, ,$$

where the left map is induced by multiplication with π^s and the right is induced by the natural quotient map $\mathcal{G}_r \to \mathcal{G}_r \otimes_{\mathfrak{o}_r} \mathfrak{o}_s \cong \mathcal{G}_s$.

[2] Called locally constant in [FK] I §12. In French one uses "lisse" or previously "constant tordue".